Area under a Curve

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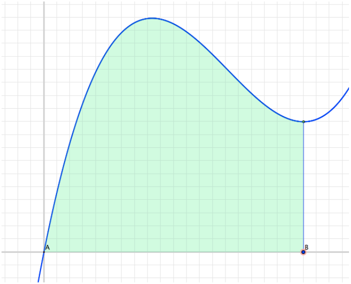
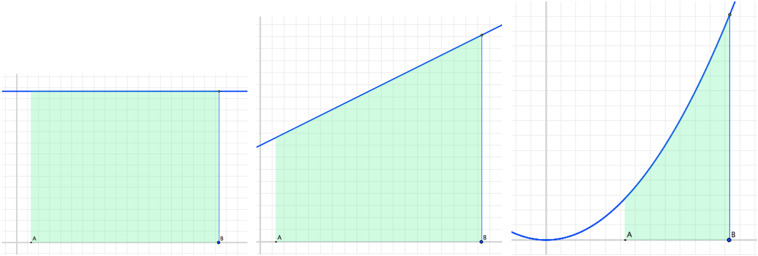
What is meant by the area under a curve, and how can its value be found?



Here are some examples.



(a) $f(x) =$ constant (b) $f(x) = \frac{1}{2}x + 1$ (c) $f(x) = x^2$

Figure: Areas under a Curve, from $x = A$ to $x = B$

In all of the above cases, we consider the area under the graph of a function, over the x-axis, and between the vertical lines $x = a$ and $x = b$.

The third case takes us into unknown territory. How should we proceed?

Suppose that $f(x)$ is a polynomial with the property that $f(x) \geq 0$ on the interval $[a, b]$. Let $F_{[a,b]}$ be the area under the graph of $f(x)$, over the x-axis, and between the vertical lines $x = a$ and $x = b$.

Exercise: Suppose $f(x) = c$ is constant on the interval $[a, b]$. Express $F_{[a,b]}$ in terms of a, b, and c.

If $f(x)$ is approximately constant on the interval $[a, b]$, then $F_{[a,b]}$ is approximately equal to $(b - a)c$.

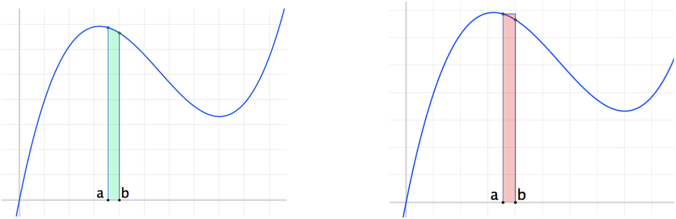


Figure: The area on the left can be approximated by the area on the right

Since the values of polynomials change only gradually, a polynomial is approximately constant on a small interval. So, given a polynomial $f(x)$ and an interval $[a, b]$, we may divide the interval $[a, b]$ into smaller subintervals on which $f(x)$ is approximately constant. We can then approximate $F_{[a,b]}$ by the areas of rectangles whose bases are the subintervals of $[a, b]$ and whose height is determined by $f(x)$.

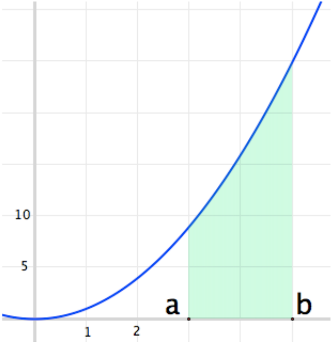


Figure: Suppose $f(x) = x^2$. We'll use $k = 4$ rectangles to approximate $F_{[3,5]}$, shown in the picture above.

$F_{[3,5]} = F^1_{[3,5]} + F^2_{[3,5]} + F^3_{[3,5]} + F^4_{[3,5]}$, as illustrated by the following picture.

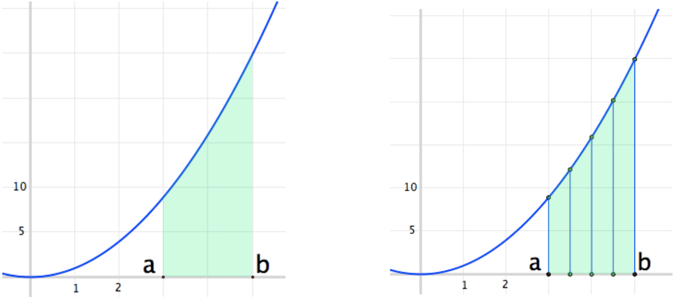
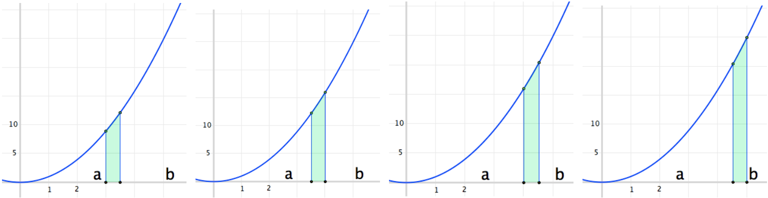


Figure: The areas on the left and the right both equal $F_{[3,5]}$

Here $F^n_{[a,b]} = F_{[a+(n-1)\Delta, a+n\Delta]}$ is a *slice*, where $\Delta = \frac{b-a}{k}$ and k is the number of subintervals.



(a) $F^1_{[3,5]}$ (b) $F^2_{[3,5]}$ (c) $F^3_{[3,5]}$ (d) $F^4_{[3,5]}$

Figure: The area $F_{[3,5]} = F^1_{[3,5]} + F^2_{[3,5]} + F^3_{[3,5]} + F^4_{[3,5]}$

We now have to choose a method by which to approximate the area of each slice. We'll use what's called the **Left Hand Rule**. In other words, we'll approximate each slice by a rectangle whose height is given by the value of $f(x)$ at the left hand endpoint of the appropriate subinterval.

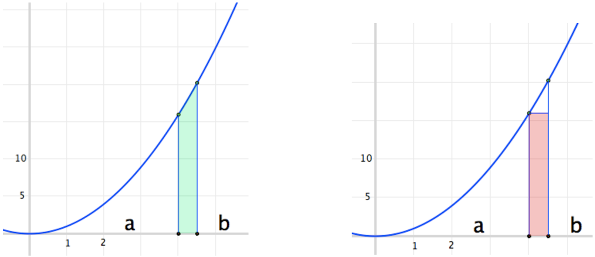


Figure: On the left, a slice. On the right, its left hand rule approximation.

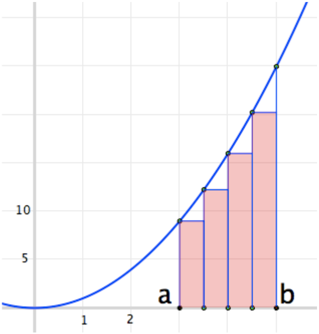


Figure: The left hand rule approximation of $F_{[3,5]}$

$$F_{[3,5]} = F^1_{[3,5]} + F^2_{[3,5]} + F^3_{[3,5]} + F^4_{[3,5]} \approx$$

$$\frac{1}{2} \times f(3) + \frac{1}{2} \times f(3.5) + \frac{1}{2} \times f(4) + \frac{1}{2} \times f(4.5)$$

$$= \frac{9+12.25+16+20.25}{2} = 28.75.$$

To build our left hand rectangles we made the height equal to the value of the function at the left end of each interval, but we did not have to choose the left endpoint. In fact we could have chosen any point in the interval. If instead we choose the right endpoint of each interval, this is the **Right Hand Rule**.

Let's redo the above example using the Right Hand Rule:

$$F_{[3,5]} = F^1_{[3,5]} + F^2_{[3,5]} + F^3_{[3,5]} + F^4_{[3,5]}$$

$$\approx \frac{1}{2} \times f(3.5) + \frac{1}{2} \times f(4) + \frac{1}{2} \times f(4.5) + \frac{1}{2} \times f(5)$$

$$= \frac{1}{2}[12.25 + 16 + 20.25 + 25] = 36.75.$$

This method of approximating the area under the curve as a sum of the areas of simpler regions lying over subintervals is called a **Riemann Sum**.

Exercise 1 Let $f(x) = 1 + \frac{1}{2}x$ and let $k = 9$. Sketch $F^3_{[2,5]}$. Then estimate the value of $F^3_{[2,5]}$, using first the Left Hand Rule and then the Right Hand Rule.

Exercise 2 Approximate $F_{[0,6]}$, where $f(x) = 5$. Use 3 rectangles and both the Left and Right Hand Rules.

Exercise 3 Approximate $G_{[1,2]}$, where $g(x) = x$. Use 4 rectangles and the Right Hand Rule.

Exercise 4 Approximate $H_{[-2,3]}$, where $h(x) = x^2 + 1$. Use 5 rectangles and the Left Hand Rule.

There are other types of Riemann sum. One could use rectangles whose heights are determined by the value of the function at the midpoints of each of the subintervals. This is called the Midpoint Rule. For example, if $f(x) = x^2$, then the Midpoint Rule Riemann sum with two subintervals which approximates $F_{[1,2]}$ is

$$F_{[1,2]} \approx \tfrac{1}{2} \cdot 1.25^2 + \tfrac{1}{2} \cdot 1.75^2.$$

Exercise 5 Approximate $J_{[1,5]}$, where $j(x) = x^3 - x$. Use 2 rectangles and the Midpoint Rule.

The Trapezoid Rule uses trapezoids instead of rectangles. So, for example, if $f(x) = x^2$, then the Trapezoid Rule Riemann sum with two subintervals which approximates $F_{[1,2]}$ is

$$F_{[1,2]} \approx \frac{1}{2} \cdot \frac{1^2+1.5^2}{2} + \frac{1}{2} \cdot \frac{1.5^2+2^2}{2}.$$

Exercise 6 Approximate $K_{[2,4]}$, where $k(x) = 2x^2 + x$. Use four subintervals and the trapezoid rule.

Let $f(x) = x$ and suppose that n is a positive integer.

What is the (Right Hand Rule) Riemann Sum approximation of $F_{[0,1]}$ with n rectangles?

The (Right Hand Rule) Riemann Sum approximation of $F_{[0,1]}$ with n rectangles, where $f(x) = x$ and $n \geq 1$, is

$$F_{[0,1]} \approx \frac{1}{n} \cdot \frac{1}{n} + \frac{1}{n} \cdot \frac{2}{n} + \frac{1}{n} \cdot \frac{3}{n} + \cdots + \frac{1}{n} \cdot \frac{n}{n} = \frac{1}{n^2}[1+2+3+\cdots+n] = \frac{1}{n^2} \sum_{k=1}^{n} k.$$

So far, we've managed to approximate the area under a curve. Is there any way to find the exact value of the area? Unsurprisingly, the answer is yes!

Once again, let $f(x) = x$. We'll use Riemann sums to find the *exact* value of $F_{[0,1]}$. The Right Hand Rule Riemann sum approximation of $F_{[0,1]}$ with n rectangles is

$$F_{[0,1]} \approx \frac{1}{n^2}(1 + 2 + \cdots + n) = \frac{1}{n^2} \cdot \frac{n(n+1)}{2} = \frac{n(n+1)}{2n^2}.$$

The Riemann sum above better approximates $F_{[0,1]}$ as n increases. Note that

$$\frac{n(n+1)}{2n^2} = \frac{n^2+n}{2n^2} = \frac{1}{2} + \frac{1}{2n},$$

which is just a transformation of $y = \frac{1}{n}$.

That means the graph of our Riemann Sum vs. n, the number of rectangles, will have the horizontal asymptote $y = \frac{1}{2}$. Consider its graph:

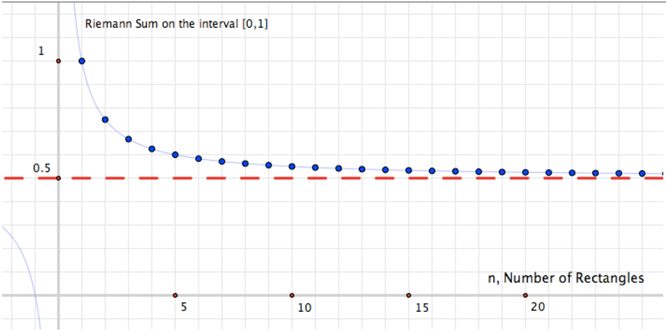


Figure: The right hand rule approximation of the area under $y = x$ on the interval $[0, 1]$, as a function of the number of rectangles

Clearly, $F_{[0,1]} = \lim_{n \to \infty} \dfrac{n(n+1)}{2n^2} = \dfrac{1}{2}$.

Postulates of Area under a Curve

We have not yet discussed $F_{[a,b]}$ for polynomials f(x) which are negative somewhere on the interval $[a, b]$. Here's the first step:

Postulate 1: Suppose that $f(x) \leq 0$ on the interval $[a, b]$, and let $g(x) = -f(x)$. Then $F_{[a,b]} = -G_{[a,b]}$.

Why does this postulate makes sense?

Suppose that c is a real number between a and b.

Postulate 2: $F_{[a,b]} = F_{[a,c]} + F_{[c,b]}$.

Why does this postulate makes sense?

Example Consider $f(x) = x - x^3$ on the interval $[0, 2]$. $f(x) \geq 0$ on $[0, 1]$ and $f(x) \leq 0$ on $[1, 2]$.

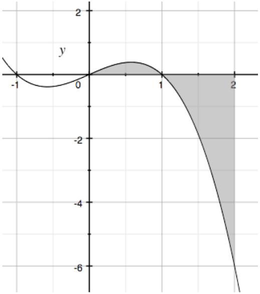
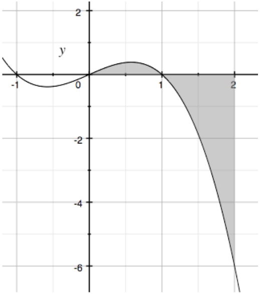


Figure: $f(x) = x - x^3$, shaded on the interval $[0, 2]$

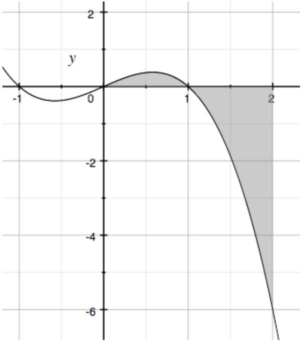
By **Postulate 2** we can break this area up into two regions: from 0 to 1 and from 1 to 2. This gives us $F_{[0,2]} = F_{[0,1]} + F_{[1,2]}$.



Let $g(x) = -f(x)$. By **Postulate 1** we have $F_{[1,2]} = -G_{[1,2]}$.

Thus, $F_{[0,2]} = F_{[0,1]} - G_{[1,2]} = F_{[0,1]} - \left|F_{[1,2]}\right|$

In other words, the area under the curve, known as the *algebraic area* or *signed area*, is the positive area above the x-axis minus the positive area below the x-axis.



Exercise From the picture above, will $F_{[0,2]}$ be positive or negative? Express the shaded area, which is not $F_{[0,2]}$, in terms of $F_{[0,1]}$ and $F_{[1,2]}$.

Postulate 3: $F_{[a,b]} = -F_{[b,a]}$.

Why does this postulate makes sense?

If $f(x) = c$ is constant, and $b < a$, then what is the value of $F_{[a,b]}$?

Accumulation Functions

Given a function $f(x)$, we can construct a new function $F(x)$ from the area under the graph of $f(x)$, as follows. Consider the function $y = f(t)$ and, for some fixed value a, define $F(x)$ by $F(x) = F_{[a,x]}$. In other words, $F(x)$ is the area under the graph of $f(t)$, from $t = a$ to $t = x$.

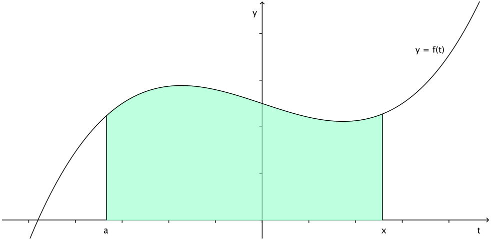


Figure: $F(x) =$ the area under $f(t)$, from $t = a$ to $t = x$

Example 1

If $f(t) = 2$ and $a = 1$, then $F(x) = F_{[1,x]} = (x-1) \cdot 2 = 2x - 2$.

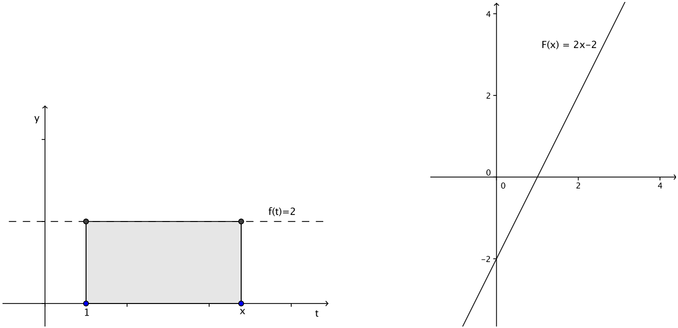
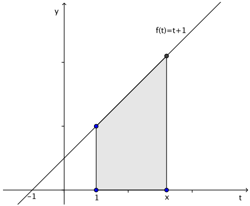


Figure: The area on the left is graphed, as a function of x, on the right

Example 2 If $f(t) = t + 1$, $a = 1$, then

$$F(x) = \frac{1}{2}(2 + x + 1)(x - 1) = \frac{1}{2}(x^2 + 2x - 3) = \frac{1}{2}(x - 1)(x + 3)$$



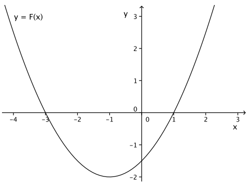


Figure: The area on the left is graphed, as a function of x, on the right

Example 3

If $f(t) = t + 1$, $a = -1$, then $F(x) = \frac{1}{2}(x+1)(x+1) = \frac{1}{2}(x+1)^2$.

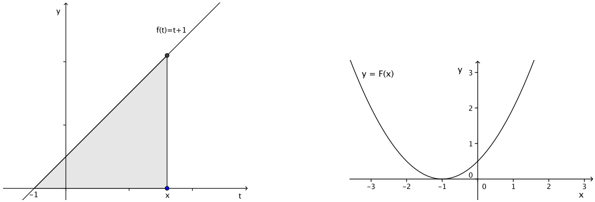


Figure: The area on the left is graphed, as a function of x, on the right

Example 4 If $f(t) = t + 1$, $a = -3$, then

$$F(x) = \frac{1}{2}(x+1)(x+1) - \frac{1}{2}(2)(2) = \frac{1}{2}(x^2+2x-3) = \frac{1}{2}(x-1)(x+3).$$

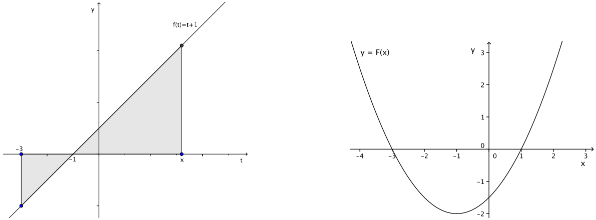


Figure: The area on the left is graphed, as a function of x, on the right